G & T Seminar Imperial College

1:30 – 2:30 Friday.

Birational Geometry

Setup Complex projective varieties.

$X \subset \mathbb{P}^N_{\mathbb{C}}$ zeroes of polynomials.

How to classify these objects?

$n = \dim X$

$\boxed{n=1}$ Alg curves / Riemann Surfaces.

3 families

Sphere		Elliptic Curves Tori 	Hyperbolic curves 
Diff geom	$\exists$ metric with positive curvature	. . . . with zero curvature.	$\exists$ metric with negative curvature
Topology	$g=0$ $\pi_1 = 1$	$g=1$ something in the middle.	$g>1$ Huge.
Arithmetic	Lots of rational pts	"	finitely many (Faltings)

Dynamics

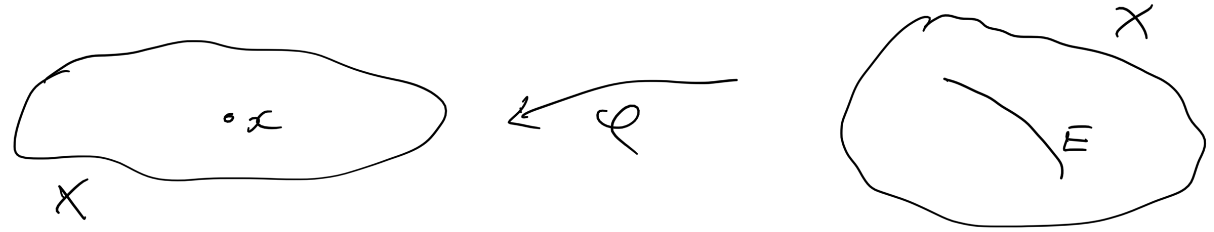
Aut Huge // finite.

$$\mathrm{Aut}(X) = \{ f : X \to X \}$$
inv + alg.

$\boxed{n>1}$?

X = smooth surface ($\dim_{\mathbb{C}} = 2$)

$$X \xleftarrow{\varphi} \tilde{X}$$



x = point $\qquad E \simeq S^2$

$$X \setminus \{x\} \underset{\text{isom.}}{\simeq} \tilde{X} \setminus E \qquad (\tilde{X} \sim_{bir} X)$$

Directed Graph

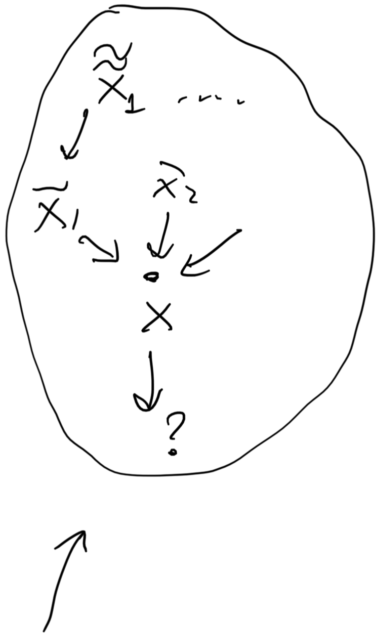
vertices

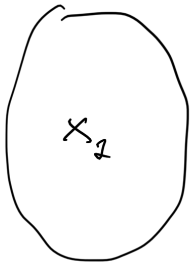
edges (oriented)

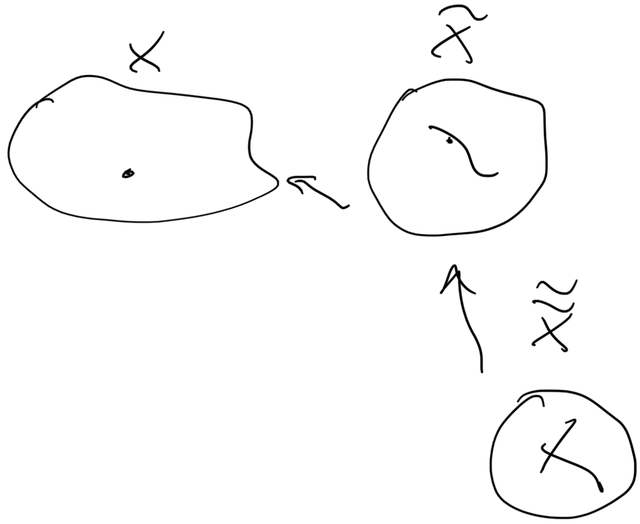
Vertices = smooth projective surfaces.

Edges = "Y = blow-up of X at one pt of X."

$$X \longleftarrow Y$$







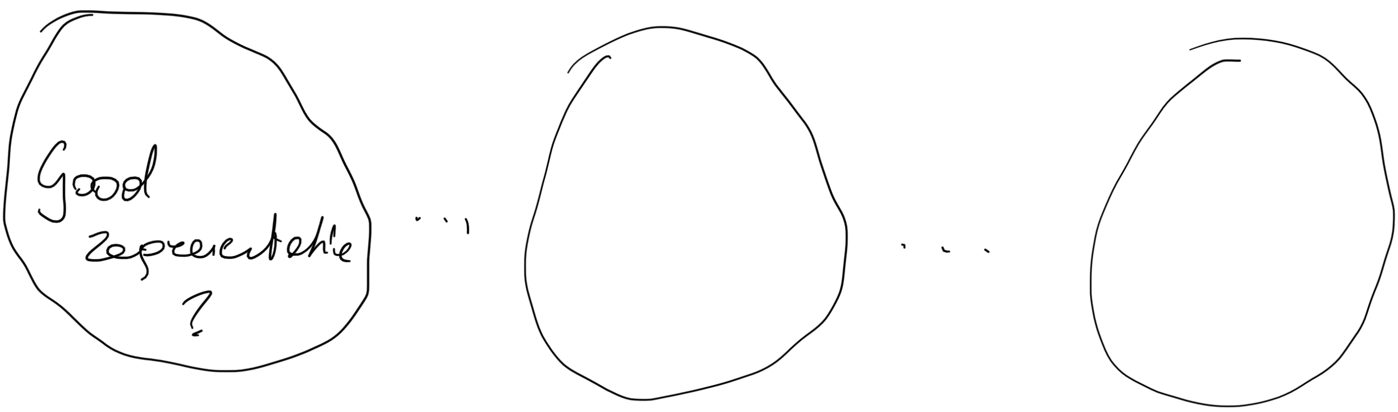
eq. class for birationality

~~Def~~ X & Y are birational if

$\exists\, Z_1 \subset X \quad Z_2 \subset Y$ closed proper

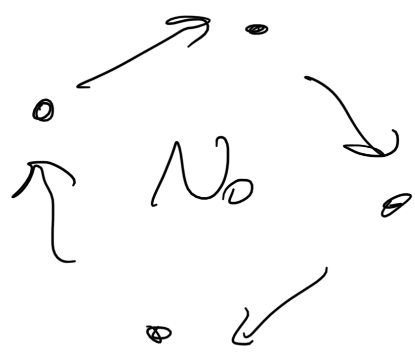
s.t. $X \setminus Z_1 \underset{\text{isom}}{\sim} Y \setminus Z_2$

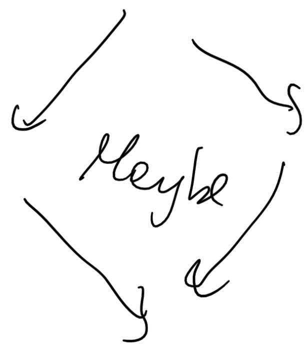
$X = \mathbb{P}^2$ set of surfaces inside the connected component of X is called the set of rational surface.



Properties (Topology)

1) $\nexists$ oriented cycle





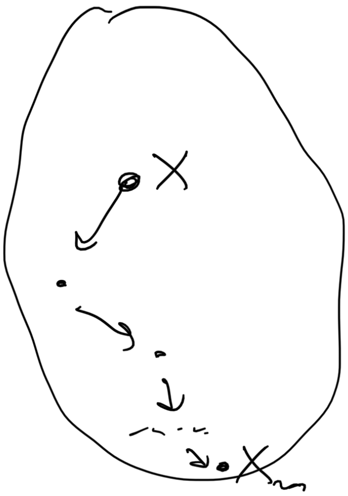
2) Starting from x, $\nexists$ infinite path of edges.

$$\overset{x}{\bullet} \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots\cdots$$ No

$$\cdots x_3 \longrightarrow x_2 \longrightarrow x_1 \longrightarrow \bullet\, x$$

<u>Theorem</u> Each connected component admits a final object, i.e.

Surface X_m s.t. $\nexists X_m \to Y$



X_m minimal model.

How many minimal model?

Depends?

<u>Unique</u> iff X is not uniruled

UNIRULED

(X uniruled if $\forall x \in X$

$$\exists\, S^2 = \mathbb{P}^1 \xrightarrow{f} X \quad \text{nonconstant}$$

s.t. $f(\mathbb{P}^1) \ni x$)

if X is uniruled $\exists$ infinitely many minimal models.

K3, Abelian.

$X \subset \mathbb{P}^N$ $\quad F_1, \dots, F_k$ polynomials.

$\deg F_i >> 0$ then X is of ~~good~~ general type.

$\boxed{n \geq 3}$ __Questions.__

How to extend this graph.

__Vertices__ are mildly singular varieties.

__Edges__ birational transformations.

$n = 3$ Mori.

$n > 3$ 2010 this directed graph exists.

__Theorem__ No oriented cycles.

__Conjecture (Termination of flips)__

$\nexists$ infinite path below a given vertex X.

Yes if $\underline{n = 3}$ Shokurov

Almost yes if X is of general type (2010)

$\exists$ path that terminates

What about if we work with

a field $K \neq \mathbb{C}$

a ring (Mixed characteristic)